Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
February 15, 2007

- 4936: *Proposed by Kenneth Korbin, New York, NY.*

  Find all prime numbers P and all positive integers a such that $P-4=a^4$.

- 4937: *Proposed by Kenneth Korbin, New York, NY.*

  Find the smallest and the largest possible perimeter of all the triangles with integer-length sides which can be inscribed in a circle with diameter 1105.

- 4938: *Proposed by Luis Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b and c be the sides of an acute triangle ABC. Prove that

$$\csc^2\frac{A}{2}+\csc^2\frac{B}{2}+\csc^2\frac{C}{2}\geq 6\left[\prod_{cyclic}\left(1+\frac{b^2}{a^2}\right)\right]^{1/3}$$

- 4939: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  For any positive integer n, prove that

$$\left\{4^n+\left[\sum_{k=0}^{n-1}3^{k+1/2}\binom{2n}{2k+1}\right]^2\right\}^{1/2}$$

  is a whole number.

- 4940: *Proposed by Michael Brozinsky, Central Islip, NY and Leo Levine, Queens, NY .*

  Let $S=\{n\in N|n\geq 5\}$. Let $G(x)$ be the fractional part of x, i.e., $G(x)=x-[x]$ where $[x]$ is the greatest integer function. Characterize those elements T of S for which the function

$$f(n) = n^2\Big(G\big(\frac{(n-2)!}{n}\big)\Big) = n.$$

- 4941: *Proposed by Tom Leong, Brooklyn, NY.*

  The numbers $1, 2, \cdots, 2006$ are randomly arranged around a circle.
  (a) Show that we can select 1000 adjacent numbers consisting of 500 even and 500 odd numbers.
  (b) Show that part (a) need not hold if the numbers were randomly arranged in a line.

Solutions

- 4900: *Proposed by Kenneth Korbin, New York, NY.*

  Find three pairs of positive integers (a, b) with $a < b$ such that triangles with sides $(a, b, 25)$ can be inscribed in a circle with diameter 65.

  Solution by David E. Manes, Oneonta, NY.

  Five such pairs of positive integers are $(16, 39), (33, 52), (39, 56), (52, 63)$, and $(60, 65)$. Assume the triangle has vertices A, B, and C with opposite sides a, b, and c respectively. Then one can argue geometrically that $\sin(\angle ACB) = \dfrac{c}{2R}$, where R is the radius of the circumscribed circle. Thus, $\sin(\angle ACB) = \dfrac{25}{65} = \dfrac{5}{13}$ so that $\cos(\angle ACB) = \pm\dfrac{12}{13}$. If $\cos(\angle ACB) = \dfrac{12}{13}$, then by the law of cosines,

  $$615 = a^2 + b^2 - \frac{24ab}{13} \quad \text{or} \quad 13\text{a}^2 - 24\text{ab} + 13\text{b}^2 - 625 \cdot 13 = 0.$$

  Note that the quadratic equation is symmetric in a and b. Solving for a, one obtains

  $$a = \frac{24b \pm 10\sqrt{4225 - b^2}}{26}. \quad \mathbf{(1)}$$

  Since a is an integer, it follows that $4225 - b^2 = x^2$ for some integer x. This equation has a finite number of solutions for b; namely $b = 16, 25, 33, 39, 52, 56, 60, 63$, or 65. Substituting the values $39, 52, 56, 63$, and 65 for b in **(1)** and using the negative sign for the square root yields the five stated solutions. Finally, if $\cos(\angle ACB) = \dfrac{-12}{13}$, then no solutions for triangles are obtained.

  Also solved by Dionne Bailey, Elsie Campbell,& Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Tom Leong, Brooklyn, NY; Peter E. Liley, Lafayette, IN, and the proposer.

- 4901: *Proposed by Kenneth Korbin, New York, NY.*

  Given pentagon $ABCDE$ with sides $\overline{AB} = 468$, $\overline{BC} = 580$, $\overline{CD} = 1183$, and $\overline{DE} = 3640$. Find the length of side $\overline{AE}$ so that the area of the pentagon is maximum.

  Solution by Tom Leong, Brooklyn, NY.

  Pentagon $ABCDE$ along with its reflection about line AE yield an octagon all of whose sides are given. Pentagon $ABCDE$ has maximum area if and only if the octagon has maximum area. It is well-known that the maximum area of a polygon with prescribed sides occurs when the polygon is inscribed in a circle (see, for example, G. Polya, *Mathematics* *and Plausible Reasoning*, Princeton University Press, 1990). Hence pentagon $ABCDE$ has maximum area when it is inscribed in a (semi)circle with AE as diameter.

Let O denote the center of this circle and put $a = \overline{AB}, b = \overline{BC}, c = \overline{CD}, d = \overline{DE}, x = \overline{AE}$ and $\vartheta_a = \angle AEB, \vartheta_b = \angle BEC, \vartheta_c = \angle CAD, \vartheta_d = \angle DAE$. The (extended) Law of Sines in triangle AEB gives $\sin\vartheta_a = a/x$ and consequently $\cos\vartheta_a = \sqrt{x^2-a^2}/x$. We obtain similar formulas for $\vartheta_b, \vartheta_c, \vartheta_d$ by looking at triangles BEC, CAD, DAE. Since $(\vartheta_a+\vartheta_b)+(\vartheta_c+\vartheta_d) = \frac{1}{2}\angle AOC + \frac{1}{2}\angle EOC = 90^\circ$, we have $\sin(\vartheta_a+\vartheta_b) = \cos(\vartheta_c+\vartheta_d)$. Thus $\sin\vartheta_a\cos\vartheta_b + \cos\vartheta_a\sin\vartheta_b = \cos\vartheta_c\cos\vartheta_d - \sin\vartheta_c\sin\vartheta_d$

$\dfrac{a}{x}\cdot\dfrac{\sqrt{x^2-b^2}}{x} + \dfrac{\sqrt{x^2-a^2}}{x}\cdot\dfrac{b}{x} = \dfrac{\sqrt{x^2-c^2}}{x}\cdot\dfrac{\sqrt{x^2-d^2}}{x} - \dfrac{c}{x}\cdot\dfrac{d}{x}$

$a\sqrt{x^2-b^2} + b\sqrt{x^2-a^2} = \sqrt{(x^2-c^2)(x^2-d^2)} - cd$. Clearing radicals, we would obtain a quartic equation in x^2 which in theory is solvable. However, using a computer algebra system is quicker and easier. Using the obvious bounds $3640 = d < x < a+b+c+d = 5871$, we obtain $x = 4225$ which can be verified as the exact answer.

Also solved by the proposer.

- 4902: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Prove that

$$F_nF_{n+1} \le \frac{2}{n+1}\sum_{k=1}^{n} kF_k^2$$

where F_k is the k^{th} Fibonacci number defined by $F_0 = 0, F_1 = 1$ and for $k \ge 2, F_k = F_{k-1} + F_{k-2}$.

Solution by Brian D. Beasely, Clinton, SC.

It is straightforward to show that the given inequality holds for $n \in \{1,2,3,4\}$. For $n \ge 5$, we prove the stronger inequality

$$F_nF_{n+1} \le \frac{2}{n+1}(nF_n^2), \quad \text{or equivalently} \quad F_{n+1} \le \frac{2n}{n+1}F_n.$$

Since $5/3 \le 2n/(n+1)$ for $n \ge 5$, it suffices to show that $F_{n+1} \le (5/3)F_n$ for $n \ge 5$. We use the Binet formula $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ for $n \ge 0$, where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Then

$$3F_{n+1} \le 5F_n \iff \frac{3(\alpha^{n+1}-\beta^{n+1})}{\sqrt{5}} \le \frac{5(\alpha^n-\beta^n)}{\sqrt{5}}$$

$$\iff (5-3\beta)\beta^n \le (5-3\alpha)\alpha^n,$$

where we note that both $5-3\beta$ and $5-3\alpha$ are positive. Since $\beta < 0 < \alpha$, this last inequality holds for odd n. For even n, it holds when

$$n \ge \frac{\log((5-3\beta)/(5-3\alpha))}{\log(\alpha/|\beta|)} = 4,$$

so we are done.

Addendum. Using the Binet formula again and noting that $\alpha > 1$ while $|\beta| < 1$, we have the corresponding asymptotic result $F_{n+1} \sim \dfrac{\alpha n}{n+1}F_n$.

Also solved by the proposer Dionne Bailey, Elsie Campbell & Charles Diminnie, San Angelo, TX; N. J. Kuenzi, Oshkosh, WI; Tom Leong, Brooklyn, NY; Carl Libis, Kingston, RI; Charles McCracken, Dayton, OH, and the proposer.

- 4903: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let n be a nonnegative integer. Prove that

$$(n!)^4(n^2+6n+11)^n \geq 2^{n+2}3^{2n+1}(n+1)^{n-3}(n+2)^{n-2}(n+3)^{n-1}$$

Solution by Paul M. Harms, North Newton, KS.

When n is a positive integer an inequality from a Stirling Formula is $n! > \sqrt{2n\pi}(n/e)^n$. Replacing the factorial in the problem by this (Stirling) inequality, it is shown below that for large enough n,

$$2^2n^2\pi^2n^{4n}e^{-4n}(n^2+6n+11)^n \geq \frac{2^n2^23^{2n}3(n+1)^n[(n+2)(n+3)]^n}{(n+1)^3(n+2)^2(n+3)}.$$

Multiplying by positive numbers, simplifying and using $n^2+6n+11=(n+3)^2+2$, the last inequality is equivalent to the following inequality:

$$(n+1)^3(n+2)^2(n+3)n^2\pi \geq [2e^43^2/n^3]^n[3/\pi][(n+1)/n]^n[(n+2)(n+3)/\{(n+3)^2+2\}]^n.$$

Note that $2e^43^2 < 1000, 3/\pi < 1, \quad (n+2)(n+3)/\{n+3)^2+2\} < 1$ and $[\mathrm{(n+1)/n}]^{\mathrm{n}}$ approaches e from below as n increases.

When n is a positive integer greater than 9,

$$(n+1)^3(n+2)^2(n+3)n^2\pi > [1000/n^3]^n(1)(3)1^n$$

$$> [2e^43^2/n^3]^n[3/\pi][(n+1)/n]^n[(n+2)(n+3)/\{(n+3)^2+2\}]^n.$$

This means that the original problem inequality holds when n is an integer greater than 9, To complete the problem show that the original problem inequality holds for $n = 0, 1, 2, \cdot\cdot, 9$.

Also solved by Tom Leong, Brooklyn, NY, and the proposer.

- 4904: *Proposed by Richard L. Francis, Cape Girardeau, MO.*

Let S be a set of positive integers such that for any element p in S which is sufficiently large, either $p-1$ or $p+1$ is composite. Such a set is called an UP-DOWN set. The set of primes is obviously in this category. Show that the set of perfect numbers, whether even or odd, is an UP-DOWN set.

Solution by Charles McCracken, Dayton, OH.

If n is odd, then $n-1$ and $n+1$ are even and hence composite.

If n is even, $n=2^{p-1}(2^p-1)$ where p is prime. Now

$$\begin{aligned} n = 2^{p-1}(2^p-1) = 2^{2p-1} - 2^{p-1} &= 2^{\text{odd}} - 2^{\text{even}} \\ &\equiv 2 - 1 \equiv 1 \equiv 1(\text{mod}3). \end{aligned}$$

Therefore $n-1 \equiv 0(\text{mod}3)$ and hence composite.

Note we exclude the case where $p = 2$ and $n = 6$ which is the one exception to the general statement.

Also solved by Charles Ashbacher, Cedar Rapids, IA; Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, & Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Jahangeer Kholdi, Portsmouth, VA; Kenneth Korbin, New York, NY; N. J. Kuenzi, Oshkosh, WI; Tom Leong, Brooklyn, NY; David E. Manes, Oneonta, NY; Boris Rays, Landover, MD; R. P. Sealy, Sackville, New Brunswick, Canada, and the proposer.

- 4905: *Proposed by Richard L. Francis, Cape Girardeau, MO.*

Consider a set S of positive integers in which the elements range over all possible numbers of digits (such as the set of repunit numbers). Such a set S is called *digitally complete.* Which of the following are digitally complete?

1. The set of factorials?
2. The set of primes?

Solution by N. J. Kuenzi, Oshkosh, WI.

1. Consider the set of factorials. For any positive integer n, let $L(n)$ be the length of the digital representation of $n!$

Examples: $3! = 6$ so $L(3) = 1$, $5! = 120$ so $L(5) = 3$, and $10! = 3,628,800$ so $L(10) = 7$.

For $n > 10$, if $m > n$ then $L(m) > L(n)$. Now $100! = 100(99!)$ and so $L(100) = L(99) + 2$. It follows that there isn't any positive integer n for which the length of the digital representation of $n!$ is $L(99) + 1$.

If you are willing to do some multiplications you can numerically verify that $L(14) = 11$ and $L(15) = 13$. So there isn't any positive integer n for which the length of the digital representation of $n!$ is 12. The set of factorials is not digitally complete.

2. Consider the set of primes. Primes less than 10 have a single digit representation. Primes between 10 and 100 have a two digit representation. In general, any prime number p between 10^{n-1} and 10^n will have a digital representation of length n.

It is known that for $x > 3$ there is at least one prime number between x and $2x - 2$. (See Beiler, Albert H. *Recreations in the Theory of Numbers: The Queen of Mathematics Entertains,* Dover Publications, Inc. 1964, p.227).

It follows from this result that there is at least one prime number between 10^{n-1} and 10^n and so there is a prime number which has digital representation of length n. The set of primes is digitally complete.

Also solved by Brian D. Beasley, Clinton, SC; Russell Euler & Jawad Sadek (jointly), Maryville, MO; Kenneth Korbin, New York, NY; Tom Leong, Brooklyn, NY; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; R. P. Sealy, Sackville, New Brunswick, Canada, and the proposer.

Late Solutions

Late solutions were received from **R.P. Sealy of Sackville, New Brunswick, Canada** to problem 4889, and from **David C. Wilson of Winston-Salem, NC** to problem 4891.