Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, SSM Problem Department, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@barak-online.net*>.

Solutions to the problems stated in this issue should be posted before
March 15, 2005

- 4846: *Proposed by Kenneth Korbin, New York, NY.*

  If $x > y > 0 > z$, solve $\begin{cases} x^3 = 3x + y, \\ y^3 = 3y + z, \\ z^3 = 3z + x. \end{cases}$

- 4847: *Proposed by Kenneth Korbin, New York, NY.*

  Find acute angles A and B such that $\begin{cases} cosA + cos3A + cos5A &= \frac{1}{2}, \\ cosB + cos2B + cos3B &= \frac{-1}{2}. \end{cases}$

- 4848: *Proposed by José Luis Díaz-Barrero Barcelona, Spain.*

  Let n be a positive integer. Show that

  $$\sum_{k=1}^{n} L_k^2 \geq \frac{(F_{2n+1} - 1)^2}{F_n F_{n+1}}$$

  where F_n and L_n are the n^{th} Fibonacci and Lucas numbers respectively.

- 4849: *Proposed by José Luis Díaz-Barrero Barcelona, Spain.*

  Let z_1, z_2, and z_3 be the affixes of the vertices of $\triangle ABC$. Prove that

  $$|z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_1|^2 \geq 2\sqrt{3}\ \mathrm{Im}(\overline{z}_1 z_2 + \overline{z}_2 z_3 + \overline{z}_3 z_1),$$

  where z_i is a complex number, and {Im} is the imaginary part of the complex number.

- 4850: *Proposed by Richard L. Francis, Cape Girardeau, MO.*

  By definition, a triangle is perfect if its side measures are positive integers and if its perimeter numerically equals its area. Show that any two perfect triangles have the same inradius.

- 4851: *Proposed by Kenneth Korbin, New York, NY.*

  Find the value of $\sum_{N=1}^{\infty} \frac{F_N}{10^{N+1}}$, where F_N is the N^{th} Fibonacci number.

Solutions

- 4812: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  If $a_1, a_2, \ldots, a_n$ are strictly positive real numbers, prove that

$$\frac{a_1}{a_2 + \sqrt[3]{a_1 a_2^2}} + \frac{a_2}{a_3 + \sqrt[3]{a_2 a_3^2}} + \ldots + \frac{a_n}{a_1 + \sqrt[3]{a_n a_1^2}} \geq \frac{n}{2}.$$

Solution by Ovidiu Furdui, Kalamazoo, MI.

Observe that the above inequality can be written as

$$\frac{\frac{a_1}{a_2}}{1 + \sqrt[3]{\frac{a_1}{a_2}}} + \cdots + \frac{\frac{a_n}{a_1}}{1 + \sqrt[3]{\frac{a_n}{a_1}}} \geq \frac{n}{2}.$$

Let us denote $\frac{a_i}{a_{i+1}} = x_i^3$; where $\frac{a_n}{a_{n+1}} = \frac{a_n}{a_1} = x_n^3$; the inequality to prove then reads:

$$\frac{x_1^3}{1+x_1} + \frac{x_2^3}{1+x_2} + \cdots + \frac{x_n^3}{1+x_n} \geq \frac{n}{2}.$$

Also observe that $x_1 + x_2 + \cdots + x_n \geq n \sqrt[n]{x_1 x_2 \cdots x_n} = n$; since $x_1 x_2 \cdots x_n = 1$.

Let $f : (0, \infty) \longrightarrow R, f(x) = \frac{x^3}{1+x}$; $f'(x) = \frac{3x^2 + 2x^3}{(1+x)^2}$; $f'(x) \geq 0 \Longrightarrow f$ is increasing on $(0, \infty)$. $f''(x) = \frac{2x^3 + 6x^2 + 6x}{(1+x)^3} \geq 0 \Longrightarrow f$ is a convex function. By Jensen's Inequality we get that

$$f(\frac{x+1+\cdots+x_n}{n}) \leq \frac{f(x_1)+\cdots+f(x_n)}{n} \Longrightarrow$$

$$f(x_1)+\cdots+f(x_n) \geq \frac{n \cdot \left(\frac{x_1+\cdots+x_n}{n}\right)^3}{1 + \left(\frac{x_1+\cdots+x_n}{n}\right)}.$$

Since f is increasing on $(0, \infty)$ and $\frac{x_1+\cdots+x_n}{n} \geq 1$, we get that $f\left(\frac{x_1+\cdots+x_n}{n}\right) \geq f(1) = \frac{1^3}{1+1} = \frac{1}{2}$. Therefore $\sum_{i=1}^{n} f(x_i) \geq \frac{n}{2}$, g.e.d.

Also solved by the proposer.

- 4813: *Proposed by Richard L. Francis, Cape Girardeau, MO.*

  Let N be a perfect number, whether even or odd. Find all such numbers for which $N-1$ is prime.

  Solution by R. P. Sealy, Sackville, New Brunswick, Canada.

  N=6 is the only solution. We note that N must be even, since 2 is the only even prime and $N = 3$ is not a perfect number. If N is an even perfect number, then $N-1$

$$\begin{aligned} &= 2^{k-1}(2^k-1)-1 \\ &= 2^{2k-1}-2^{k-1}-1 \\ &= (2^k+1)(2^{k-1}-1) \end{aligned}$$

  which is a composite number for $k > 2$.

  Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY; Harry Sedinger, St. Bonaventure, NY; Sloane Springer, San Angelo, TX, and the proposer.

- 4814: *Proposed by Mark Applebaum and Peter Samovol, Beer-Sheva, Israel.*

  Find functions f and g such that $f : R \to R$ and $g : R \to R$ and

$$\forall x : \begin{cases} f\Big(g(x)\Big) = x^2 \\ g\Big(f(x)\Big) = x^3 \end{cases}$$

  Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX.

  If f and g have the given properties, then

$$\begin{aligned} f(x_1) = f(x_2) \quad &\Rightarrow \quad g\Big(f(x_1)\Big) = g\Big(f(x_2)\Big) \\ &\Rightarrow \quad x_1^3 = x_2^3 \\ &\Rightarrow \quad x_1 = x_2 \end{aligned}$$

  and hence, f must be injective.
  On the other hand, for all $x \in R$,

$$\Big(f(x)\Big)^2 = f\Big(g(f(x))\Big) = f(x^3),$$

  which implies that

$$\Big(f(0)\Big)^2 = f(0), \quad \Big(f(1)\Big)^2 = f(1), \text{ and } \Big(\mathrm{f}(-1)\Big)^2 = \mathrm{f}(-1).$$

  This means that $f(0), f(1), f(-1) \in \{0, 1\}$, which makes it impossible for f to be injective. Therefore, no such functions exist.

  Also solved by José Luis Díaz-Barrero, Barcelona, Spain, and the proposers.

- 4815: *Proposed by Kenneth Korbin, New York, NY.*

  Solve:

$$\begin{cases} x^2 + y^2 = (2005)^2 \\ x^3 + y^3 = (2005)^3 \end{cases}$$

Solution by David E. Manes, Oneonta, NY.

The solutions written as ordered pairs (x, y) are: $(2005, 0)$, $(0, 2005)$,

$$\left(2005\left(\frac{-2+\sqrt{2}i}{2}\right), -2005\left(\frac{2+\sqrt{2}i}{2}\right)\right), \text{ and } \left(-2005\left(\frac{2+\sqrt{2}\mathrm{i}}{2}\right), 2005\left(\frac{-2+\sqrt{2}\mathrm{i}}{2}\right)\right).$$

More generally, for any real number a, consider the system of equations

$$\begin{aligned} x^2 + y^2 &= a^2 \\ x^3 + y^3 &= a^3. \end{aligned}$$

Then $(a, 0)$ and $(0, a)$ are obvious solutions. Therefore, assume $x \neq 0$ and $y \neq 0$ and note that $x \neq -a$. Then the two equations reduce to $a - x = \dfrac{y^2}{a+x}$ and $a - x = \dfrac{y^3}{(a^2+ax+x^2)}$, so that $y = \dfrac{a^2+ax+x^2}{a+x}$. Thus, $\mathrm{x}^2 + \left(\dfrac{\mathrm{a}^2+\mathrm{ax}+\mathrm{x}^2}{\mathrm{a}+\mathrm{x}}\right)^2 = \mathrm{a}^2$, and $x \neq -a$ implies that this equation can be reduced to $2x^4 + 4ax^3 + 3a^2x^2 = 0$ or, $2x^2 + 4ax + 3a^2 = 0$ since $x \neq 0$. Therefore

$$x = a\left(\frac{-2 \pm \sqrt{2}i}{2}\right) \text{ and } \mathrm{y} = \mathrm{a}\left(\frac{-2 \mp \sqrt{2}\mathrm{i}}{2}\right).$$

The solutions to the proposed problem are then obtained by letting $a = 2005$.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; José Luis Díaz-Barrero, Barcelona, Spain; John J. Boncek, Montgomery, AL; Charles R. Garner, Jr., Conyers, GA; Paul M. Harms, North Newton, KS; Jahangeer Kholdi, Christopher Dann, and Vincent Liverman (jointly), Portsmouth, VA; Peter E. Liley, Lafayette, IN; Charles McCracken, Dayton, OH; John Nord, Spokane, WA; Harry Sedinger, St. Bonaventure, NY, and the proposer.

- 4816: *Proposed by Kenneth Korbin, New York, NY.*

  Find the value of $x^4 + y^4 + z^4$ if

$$\begin{cases} x + y + z = 2 \\ x^2 + y^2 + z^2 = 6 \\ x^3 + y^3 + z^3 = 8 \end{cases}$$

Solution by Charles R. Garner, Jr., Conyers, GA.

Solve the first equation for z and substitute into the second and third equations. This results in the system

$$\begin{aligned} 2x^2 + x(2y-4) + 2y^2 - 4y + 4 &= 6 \\ 3x^2(2-y) - 3x(y-2)^2 + 6y^2 - 12y + 8 &= 8 \end{aligned}$$

Since the first equation in this system is quadratic in x, we use the quadratic formula to get $x = \frac{1}{2}\left[2-y\pm\sqrt{4(y-2)-3y^2}\,\right]$, which we then substitute into the second equation in the system. This reduces that equation to $3y^3-6y^2-3y+14=8$. This is easily solved by setting it equal to zero and factoring: $3(y-1)(y^2-1)=0$, giving $y=-1, y=1$ or $y=2$. This results in 6 solution triples: $(1,-1,2)$, $(1,2,-1)$, $(-1,1,2)$, $(-1,2,1)$, $(2,1,-1)$, and $(2,-1,1)$. In all cases, $x^4+y^4+z^4=18$.

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell and Dionne T. Bailey (jointly), San Angelo, TX; José Luis Díaz-Barrero, Barcelona, Spain; John J. Boncek, Montgomery, AL; Paul M. Harms, North Newton, KS; Harvey Johnson and Charles Diminnie (jointly), San Angelo, TX; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; David C. Wilson, Winston-Salem, NC, and the proposer.

- 4817: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Evaluate the sums

$$a)\ \sum_{k=1}^{\infty}\frac{1+T_k}{T_k^2} \qquad\qquad b)\ \sum_{k=1}^{\infty}\frac{1}{T_kT_{k+1}}$$

where $T_k = \frac{k(k+1)}{2}$ represents the k^{th} triangular number.

Solution by Brian D. Beasley, Clinton, SC.

(a) We have

$$\sum_{k=1}^{\infty}\frac{1}{T_k} = \sum_{k=1}^{\infty}\left(\frac{2}{k}-\frac{2}{k+1}\right) = 2$$

and

$$\sum_{k=1}^{\infty}\frac{1}{T_k^2} = 4\sum_{k=1}^{\infty}\left[\frac{1}{k^2}+\frac{3}{(k+1)^2}-\left(\frac{2k}{k^2}-\frac{2k}{(k+1)^2}\right)\right].$$

Using the well-known formula $\sum_{k=1}^{\infty}(1/k^2)=\pi^2/6$, we have

$$\sum_{k=1}^{\infty}\left(\frac{2k}{k^2}-\frac{2k}{(k+1)^2}\right) = \frac{2}{1^2}-\frac{2}{2^2}+\frac{4}{2^2}-\frac{4}{3^2}+\cdots = \frac{2}{1^2}+\frac{2}{2^2}+\cdots = \frac{\pi^2}{3}.$$

Hence we conclude

$$\sum_{k=1}^{\infty}\frac{1}{T_k}+\sum_{k=1}^{\infty}\frac{1}{T_k^2} = 2+4\left[\frac{\pi^2}{6}+\left(\frac{\pi^2}{2}-3\right)-\frac{\pi^2}{3}\right] = \frac{4\pi^2}{3}-10.$$

(b) We have

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{T_kT_{k+1}} &= \sum_{k=1}^{\infty}\left(\frac{4}{k(k+2)}-\frac{4}{(k+1)^2}\right) = \sum_{k=1}^{\infty}\left(\frac{2}{k}-\frac{2}{k+2}\right)-4\left(\frac{\pi^2}{6}-1\right)\\ &= 3-\frac{2\pi^2}{3}+4 = 7-\frac{2\pi^2}{3}.\end{aligned}$$

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX; John J. Boncek, Montgomery, AL; Paul M. Harms, North Newton,

KS; David E. Manes, Oneonta, NY; David C. Wilson, Winston-Salem, NC, and the proposer.

Late Solutions

A late solution to 4810 was received from the **Churchland High School Math Club of Portsmouth, VA.**